Title: Note on additive functions of several variable

Author: Marek Kuczma

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MAREK KUCZMA

Note on additive functions of several variables

Let $f : R^2 \to R$ be an additive function of two [1]) real variables, i. e. we assume that the relation

$$(1) \qquad f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2)$$

holds for all $x_1, x_2, y_1, y_2 \in R$. As is well known [1], f can be represented then in the form

$$(2) \qquad f(x, y) = g(x) + h(y),$$

where $g : R \to R$ and $h : R \to R$ are additive:

$$(3) \qquad g(x_1 + x_2) = g(x_1) + g(x_2), \quad h(y_1 + y_2) = h(y_1) + h(y_2).$$

Decomposition (2) results immediately from (1) on setting $x_1 = x$, $y_2 = y$, $x_2 = y_1 = 0$ and $g(x) = f(x, 0)$, $h(y) = f(0, y)$.

The same argument is valid for adidtive functions $f : A \times B \to C$, where A, B, C are commutative groups (written additively). However, if A, B, C are commutative semigroups, this reasoning fails, since we cannot put $x_2 = y_1 = 0$. It is the purpose of the present note to establish whether in such a more general case a decomposition as in (2) is possible.

THEOREM 1. *Let A, B be commutative semigroups and C a commutative group. If $f : A \times B \to C$ is additive, then it can be written in form (2), where $g : A \to C$ and $h : B \to C$ satisfy (3). Decomposition (2) is unique.*

P r o o f. Fix $a \in A$, $b \in B$ and write

$$(4) \qquad g(x) = f(x + a, b) - f(a, b), \quad h(y) = f(a, y + b) - f(a, b).$$

Since f is additive, we have

[1]) The extension of the results of the present note to the case of n variables is obvious.

$$f(x, y) + f(a, b) + f(a, b) = f(x + a + a, y + b + b) = \\ = f(x + a, b) + f(a, y + b),$$

which in view of (4) implies (2). Similarly

$$f(x_1 + x_2 + a, b) + f(a, b) + f(a, b) = \\ = f(x_1 + a, b) + f(x_2 + a, b) + f(a, b),$$

which in view of (4) implies the first relation of (3). The other one is established analogously.

Supposing that we have two decompositions:

$$f(x, y) = g_1(x) + h_1(y) = g_2(x) + h_2(y)$$

with additive g_i, h_i, we obtain

$$g_1(x) - g_2(x) = h_2(y) - h_1(y),$$

i. e. $g = g_1 - g_2$ as well as $h = h_1 - h_2$ must be constant; on the other hand, g and h are additive. But relations (3) imply that the constants must be zero, i. e. $g_1 = g_2$, $h_1 = h_2$, and decomposition (2) is unique.

Now, if C is a semigroup, not a group, we cannot form functions (4). However, since every commutative semigroup in which the cancellation law holds [2]) can be embedded in a commutative group [2], Theorem 1 yields the following result.

THEOREM 2. *Let A, B, C be commutative semigroups and suppose that the cancellation law holds in C. Further, let G be a commutative group extension of C. If $f : A \times B \to C$ is additive, then there exist unique additive functions $g : A \to G$ and $h : B \to G$ such that relation (2) holds.*

The above theorem shows that in the case where A, B, C are semigroups, decomposition (2) with $g : A \to C$ and $h : B \to C$ is not always possible. For example, take $A = B = C = (1, \infty)$, $G = (-\infty, +\infty)$ (with the usual addition as the operation) and $f(x, y) = \frac{1}{2}(x + y)$. Then $g(x) = \frac{1}{2}x$, and $h(y) = \frac{1}{2}y$ (recall the uniqueness statement), but these are not functions with values in $C = (1, \infty)$.

Finally let us note the following corollary to Theorem 2.

CORALLARY. *Let A, B, C be commutative semigroups and suppose that the cancellation law holds in C. If $f : A \times B \to C$ is additive and if there exists a decomposition (2) with additive $g : A \to C$ and $h : B \to C$, then this decomposition is unique.*

[2]) I. e. $c_1 + c_0 = c_2 + c_0$ implies $c_1 = c_2$.

It would be of interest to determine what can happen in the case where C is a commutative semigroup without the concellation law, and in particular whether the above Corollary remains true in such a case.

REFERENCES

[1] J. Aczél: Lectures on functional equations and their applications, Academic Press, New York and London, 1966.

[2] A. G. Kurosz: Algebra ogólna, P. W. N., Warszawa, 1965.

MAREK KUCZMA

NOTA O FUNKCJACH ADDYTYWNYCH WIELU ZMIENNYCH

Streszczenie

W pracy bada się możliwość rozkładu

$$f(x, y) = g(x) + h(y)$$

dla addytywnej funkcji $f: A \times B \to C$, tzn. spełniającej równanie

$$f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2),$$

gdzie $g: A \to C$ i $h: B \to C$ są addytywnymi funkcjami jednej zmiennej, zaś A, B, C są przemiennymi półgrupami.

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